Exotic embeddings of cubes

joint work with Joé Brendel and Felix Schlenk

A symplectic cube $C(a)$

$D^2(a)$ = the disk of area $a>0$

$$C^{2n}(a) = \underbrace{D^2(a) \times D^2(a) \times \ldots \times D^2(a)}_{n \text{ times}} \quad \text{cube}$$

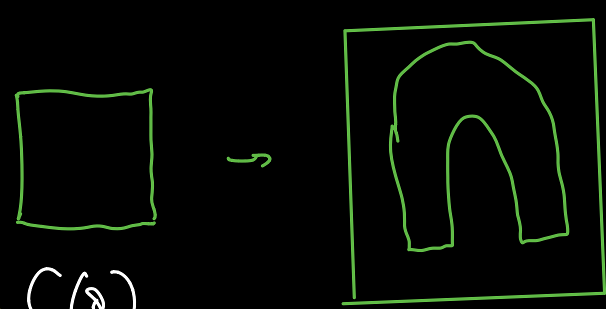
$$C^n(a) \hookrightarrow C^n(1) \quad \text{if } a<1$$

in many ways? Yes if $\frac{1}{2} < a < \frac{2}{3}$

essentially different

"

non-Hamiltonian isotopic after a coordinate permutation



$C(a)$ $\qquad$ $C(1)$

($a>\frac{1}{2}$ non-isotopic by Floer-Hofer-Wysocki)

$a<\frac{1}{2}$ isotopic to standard $\qquad$ $a>\frac{2}{3}$ too little space to embed

There are some further previous results on non-isotopic embeddings by

Hind, Gutt-Usher and Dimitroglu-Rizell

<u>Theorem:</u> (Brendel-Schlenk-M)

# of non-isotopic embeddings $C(a) \to C(1)$ grows arbitrarily large when $a \to \frac{1}{2}+$

$\frac{1}{2} < a < \frac{2}{3}$ at least two embeddings $(1,1,1)$

$\frac{1}{2} < a < \frac{6}{11}$ at least three embeddings $(1,1,3)$

- - - -

A version of the same theorem for $B^4(1)$

\# of non-isotopic embeddings $C(a) \to B^4(1)$

grows arbitrarily large when $a \to \frac{1}{3}^+$

$B^4(1) \times D^2(1) \times D^2(1) \dots$

$\frac{1}{3} < a < \frac{1}{2}$ at least one $(1,1,1)$

$\frac{1}{3} < a < \frac{6}{15}$ at least two $(1,1,2)$

$\frac{1}{3} < a < \frac{30}{87}$ at least three $(1,5,2)$

...

Furthermore, a similar theorem for embeddings to closed monotone Del Pezzo surfaces

In particular, if the degree $(=K^2)$ is $9, 8, 6, 5$ then # grows arbitrarily large for $Q \to \frac{1}{\sqrt{K^2}}^+$

$Q \to \frac{1}{3}^+$ $\quad K^2 = 9 \quad \mathbb{CP}^2$

$Q \to \frac{1}{2\sqrt{2}}^+$ $\quad K^2 = 8$ $\quad$ <u>here</u> $\mathbb{CP}^1 \times \mathbb{CP}^1 = S^2 \times S^2$

$\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ does not admit Pic $= 1$ ($=$ triangular) degenerations

$Q \to \frac{1}{\sqrt{6}}^+$ $\quad K^2 = 6 \quad \mathbb{CP}^2$ blown up 3 times

$Q \to \frac{1}{\sqrt{5}}^+$ $\quad K^2 = 5 \quad \mathbb{CP}^2$ blown up 4 times

less well-organized embeddings (probably also with their number growing arbitrarily large) for other K^2

from degenerations to Pic > 1 toric surfaces

(i.e. $\mathbb{CP}^2$ blown up $1, 2, 5, 6, 7$ and 8 times)

Markov's and Markov-type equations and their toric geometry

Markov 1879

$$a^2 + b^2 + c^2 = 3abc \qquad a,b,c \in \mathbb{N}$$

(numerology of $\mathbb{CP}^2$) such (a,b,c) called Markov's triple

Hacking–Prokhorov (advancing an earlier work of Manetti) 2010

classified projective toric surfaces with $\mathrm{Pic} = 1$

and having only T-singularities, a.k.a. Wahl

(= $\mathbb{Q}$-Gorenstein smoothable toric singularities)

and noticed that they are smoothable

globally (vanishing local-to-global obstructions)

Answer: $\mathbb{P}(a^2, b^2, c^2)$ where (a,b,c) is a Markov triple

$(1,1,1), (1,1,2), (1,5,2), \ldots$

(other more constructive proofs are also available by now)

Why to care?

A smoothing is a projective family

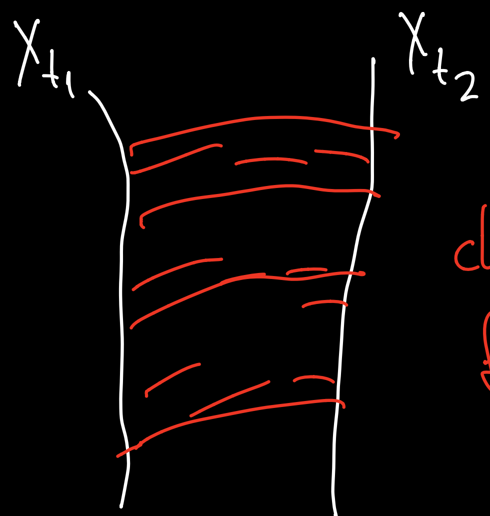
$$\begin{tikzcd} X_0 \arrow[d] & \subset & \mathcal{X} \arrow[d, "f"'] & \supset & X_t \arrow[d] \\ 0 & \in & \Delta & \ni & t \end{tikzcd}$$

singular   smooth

3-fold

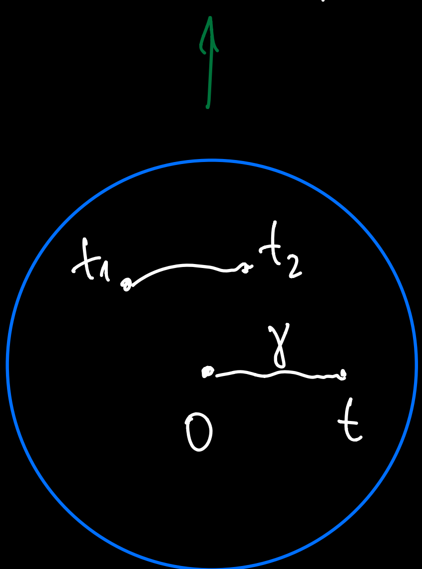
$$\begin{tikzcd} \mathcal{X} \arrow[d, "f"'] & \subset \mathbb{CP}^N \times \Delta \\ \Delta & \end{tikzcd}$$

$\Delta \leftarrow$ unit $\mathbb{C}$ disk



$X_{t_1} \cong X_{t_2}$ symplectomorphism

characteristic foliation in $f^{-1}(\gamma)$



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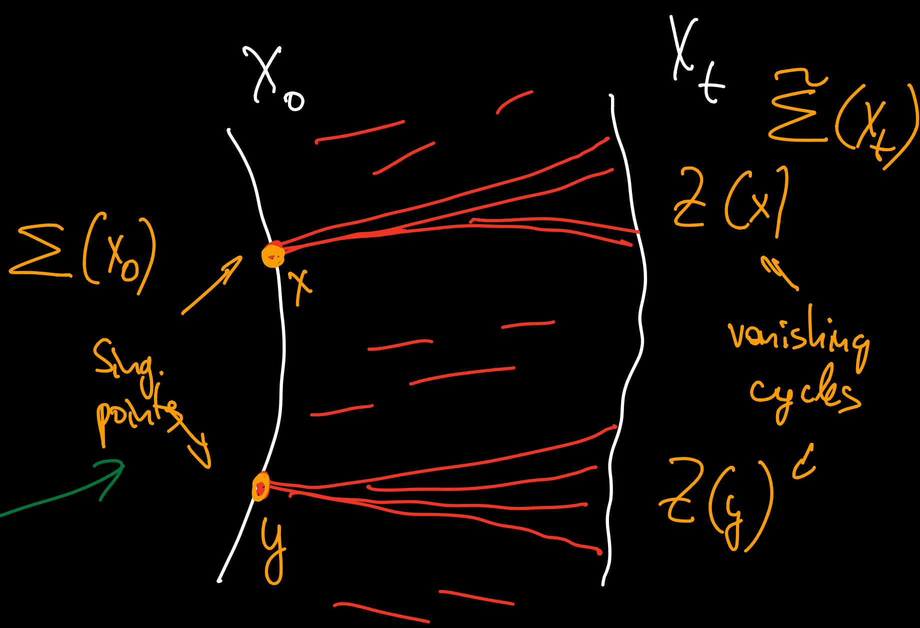


$$\underbrace{X_0^\circ}_{\text{Smooth part}} \stackrel{\text{def}}{=} X_0 \setminus \underbrace{\Sigma(X_0)}_{\text{singular locus}} \overset{\text{symplectomorphic}}{\cong} X_t \setminus \underbrace{\Sigma_\gamma(X_t)}_{\text{locus of vanishing cycles}}$$

Anything embedded to the smooth locus of X_0 gets symplectically transported to X_t !

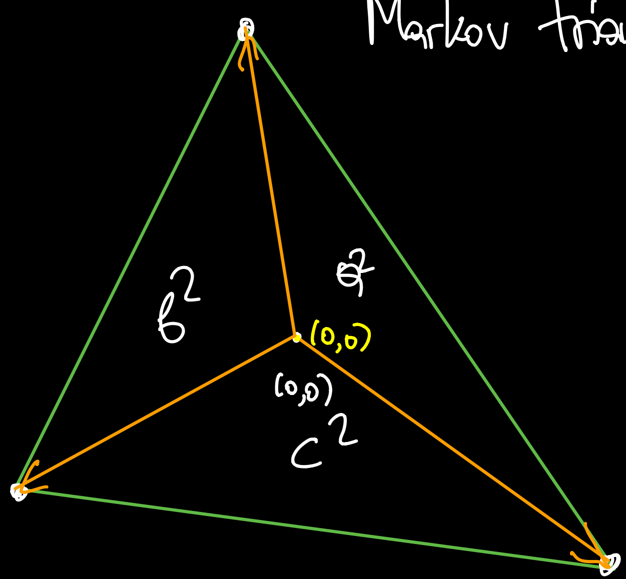
The idea of using Markov triples to study of <u>symplectic</u> geometry of $\mathbb{CP}^2$ goes back to 2010 preprint (IPMU) of Galkin and Usnich

Galkin and Usnich have upgraded Markov triples (a,b,c) to Markov lattice triangles $T_{a,b,c} \subset \mathbb{R}^2$ (corresponding to toric fans of $\mathbb{P}(a^2,b^2,c^2)$) and introduced their <u>mutations</u>

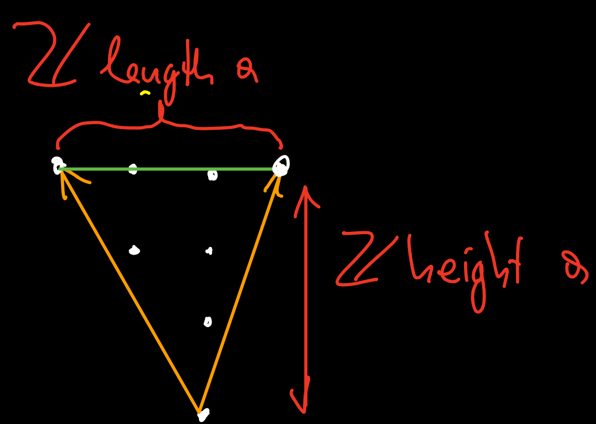
$$(a,b,c) \to (a,b,3ab-c)$$

Markov triangle

$$a^2+b^2+c^2 = 3abc$$



$\leftarrow$ a geometric representation of Markov's equation

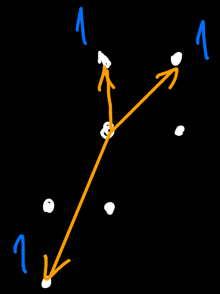


normalized area a^2

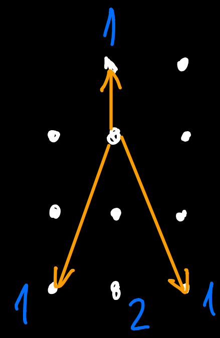
Each cone is a T_1-singularity (of Milnor number 0)

$\Vert$

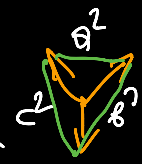
$$\frac{\text{length}}{\text{height}} \sim 1$$



$\mathbb{P}(1,1,1)=\mathbb{CP}^2$



$\mathbb{P}(1,1,4)$

$T_{a,b,c}$ = Convex Hull ()

Using toric fan mutations corresponding to Markov's mutations $(a,b,c)\mapsto(a,b,3ab-c)$, G-U have introduced an integer functions on the lattice points of T_{abc}

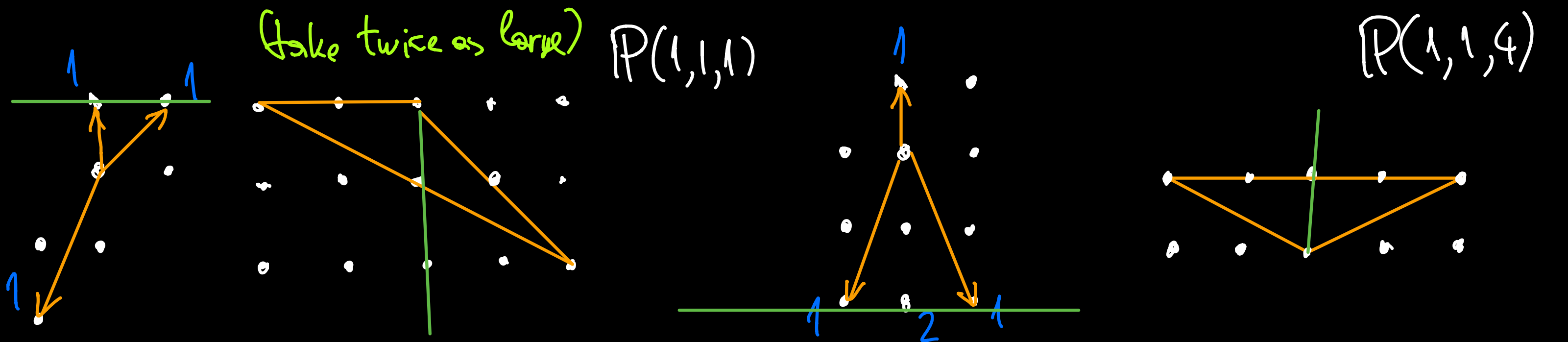
Conjecture (G-U 2010): This function is the counting Maslov 2 disk function for some corresponding Lagrangian tori

$$L_{a,b,c}\subset\mathbb{CP}^2$$

(proved by Vianna, Pascaleff-Tonkonog, Dimitroglu-Rizell-Ekholm-Tonkonog, et al.)

Let us construct $L_{a,b,c}$ directly from smoothings of $\mathbb{P}(a^2, b^2, c^2)$ (Galkin–M, unpublished)

Symplectically, $\mathbb{P}(a^2, b^2, c^2)$ is given by a dual ($N = M^*$ lattice) triangle $\Delta_{a,b,c} \subset N \otimes \mathbb{R} \simeq \mathbb{R}^2$ to $T_{a,b,c} \subset M \otimes \mathbb{R} \simeq \mathbb{R}^2$

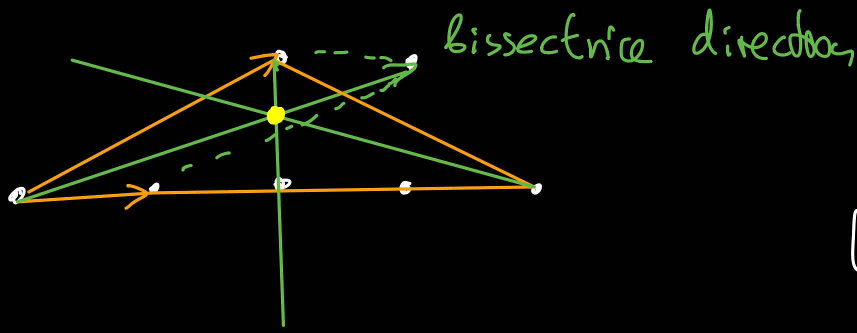


G–U mutations and dual mutations (cf. almost toric mutations of Casals–Vianna)

In this polygon we consider

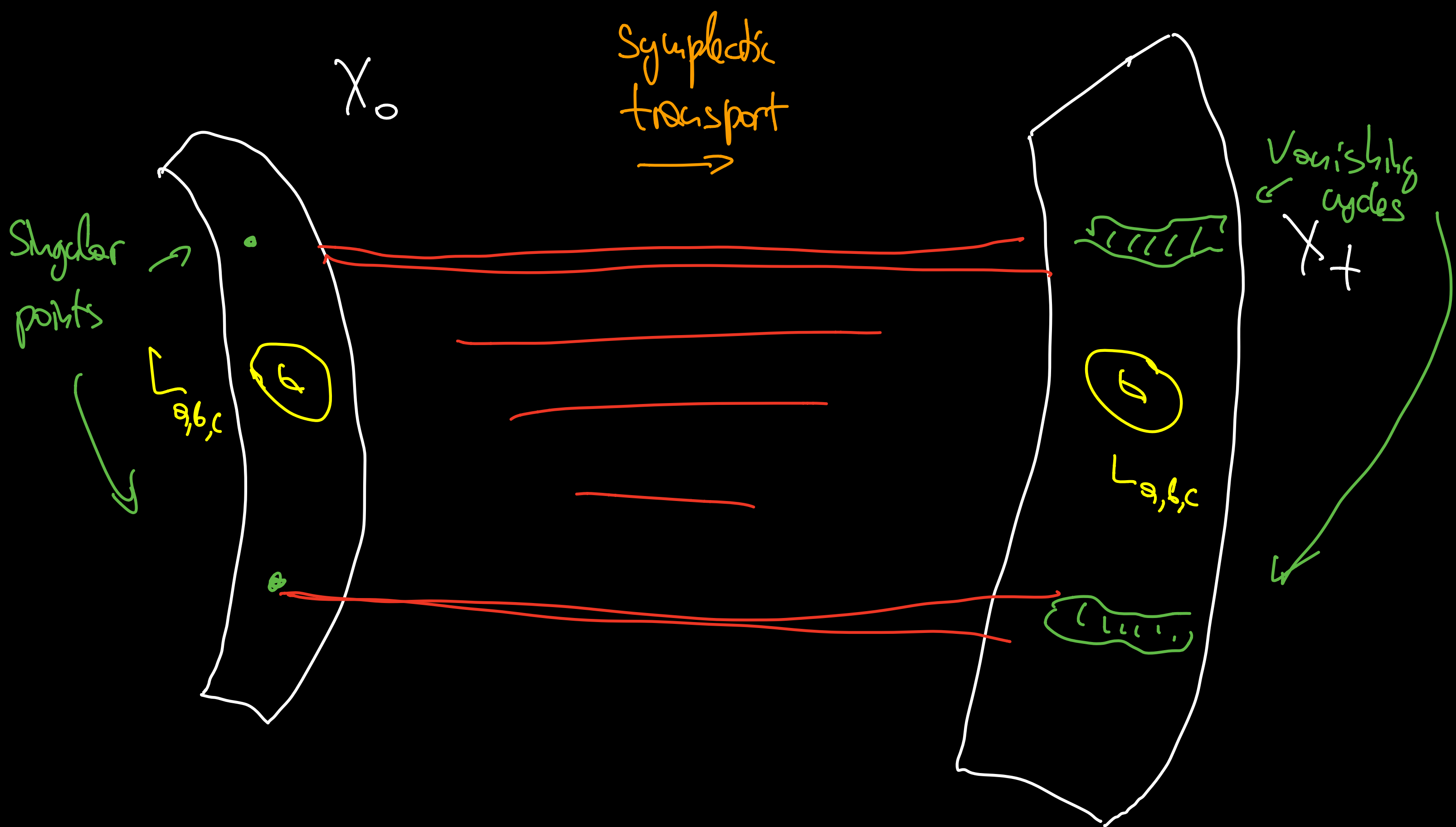
<u>integer bissectrices</u>, they intersect

in one point (Note: it is not generally the barycenter of $\Delta_{a,b,c}$)



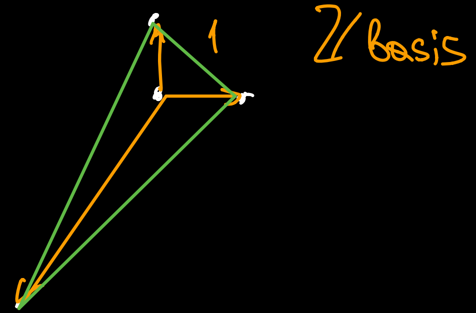
The fiber over • is a monotone Lagrangian torus contained in the smooth locus of $\mathbb{P}(a^2, b^2, c^2)$

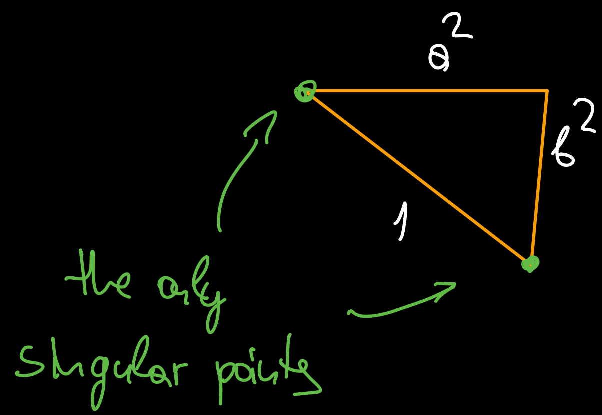
$$\Delta_{a,b,c} \subset N \otimes \mathbb{R} \simeq \mathbb{R}^2$$



Similarly, if $c=1$ then

$T_{a,b,c}$





removing the hypothenuse gives $E(a^2,b^2)$ in the smooth locus of $\mathbb{P}(a^2,b^2,1)$

$\Rightarrow$ $E(a^2,b^2)$ embeds to $\mathbb{P}(1,1,1)$ (of volume a^2b^2 = area of K $3abc$)

the hypothenuse taken with multiplicity ab is the limit of a family of lines in the approximating $\mathbb{P}^2$ $\Rightarrow$

$E(a^2,b^2)$ embeds to $B(ab)$

sharp embedding!

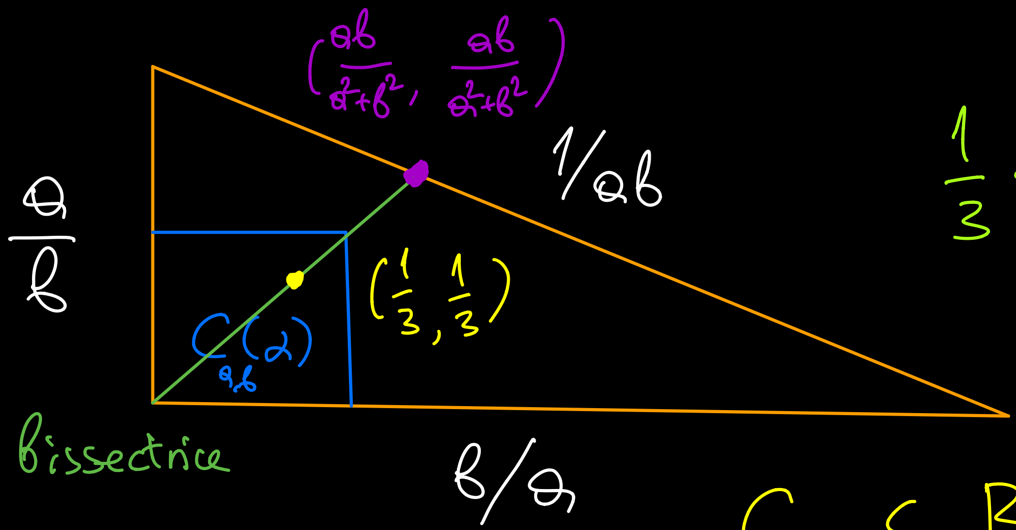
$E(a,b)=\{\frac{|z_1|^2}{a}+\frac{|z_2|^2}{b}\le 2\pi\}$

or $E(\frac{a}{b},\frac{b}{a})$ embeds to $B(1)$ (cf. Casals – Vianna)

if $(a,b,1)$ is

a Markov triple

But cubes are also in $E(a^2, b^2)$!



$$\frac{1}{3} < \alpha < \frac{ab}{a^2+b^2}$$

$C_{a,b} \subset B(1)$ by symplectic transport

for different Markov triples $(a,b,1)$

= consecutive odd Fibonacci doubles (a,b)

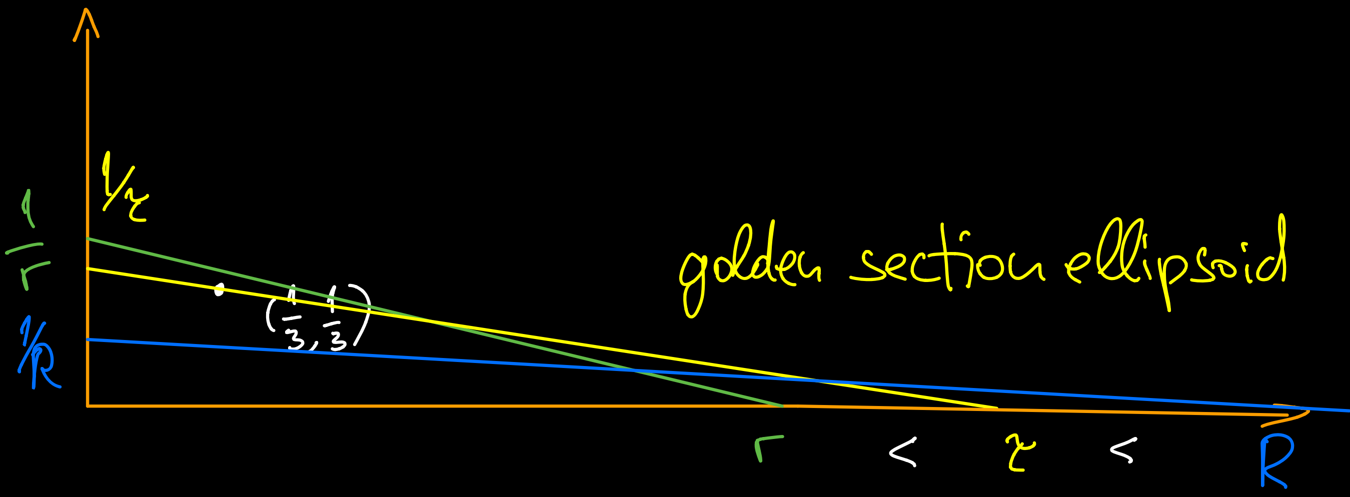
$C_{a,b}(\alpha)$ are not Hamiltonian isotopic

since $C_{a,b}(\alpha) \supset \mu^{-1}\left(\frac{1}{3}, \frac{1}{3}\right) = L_{a,b,1} \subset \mathbb{C}P^2$

$\Rightarrow$ already the restrictions to $\mu^{-1}\left(\frac{1}{3}, \frac{1}{3}\right)$ can not be Hamiltonian isotopic !

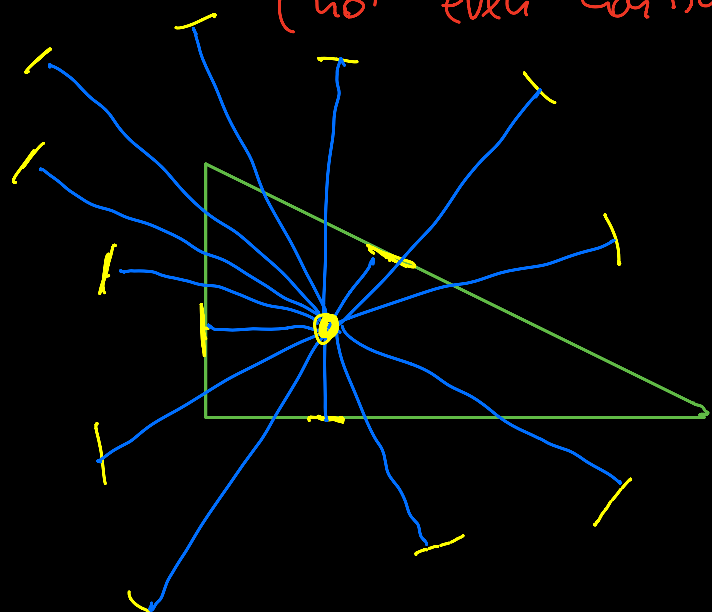
The argument survives stabilization by taking products with $B^2(1)$ (or other monotone mfds containing cubes). $\square$

We finish $C^4(\alpha) \hookrightarrow B^4(1)$ case by consideration of the following representation of the golden section:



$(\frac{1}{3}, \frac{1}{3})$ is the integer center of inscribed circle

(not even continuous)



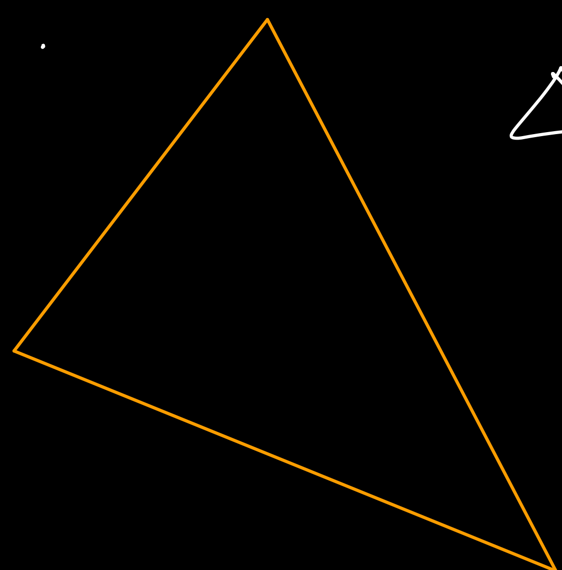
$\nexists$ inscribed circle = the bissectrice intersection point.

Cubes (sources) can be traded for other domains in X_0^0

How about targets?

$\mathbb{CP}^2$ is the Del Pezzo surface of highest possible degree (9), but there are smaller degree DPs, and they have their own Markov-type equations.

perimeter Π
area A



$\Delta \subset N \otimes \mathbb{R} \cong \mathbb{R}^2$ a lattice triangle gives $X(\Delta) \subset \mathbb{CP}^N$, Kähler toric variety

$\omega^2 = A \quad , -\omega \cdot K = \Pi$

$\omega \in H^2(X(\Delta); \mathbb{Z})$

$K \in H^2(X(\Delta); \mathbb{Q})$
$\quad\; \shortparallel$
$\quad\; \mathbb{Q}$

$$\Rightarrow \frac{-K}{\omega} = \frac{\Pi}{A}$$

$\omega \sim K$
proportional

Noether's formula

$$K^2 + \sum_{j=1}^{3} m_j = 12$$

Milnor numbers at corners + 1

$\Rightarrow$ for $K^2 < 9$ may have T_{m+1}-singularities

$\Pi = m_a a^2 + m_b b^2 + m_c c^2$

$A = m_a m_b m_c \, a^2 b^2 c^2$
$\shortparallel$
ω^2

Markov-type equations

$$\Rightarrow K^2 = \frac{\Pi}{A/\Pi} = \frac{\Pi^2}{A} = \frac{(m_a a^2 + m_b b^2 + m_c c^2)^2}{m_a m_b m_c \, a^2 b^2 c^2}$$

$$m_a a^2 + m_b b^2 + m_c c^2 = \sqrt{m_a m_b m_c (12 - m_a - m_b - m_c)}\, abc$$

must be integer

<u>Hacking - Prokhorov:</u> 14 Markov type equations

But only 4 of them

$a^2+b^2+c^2 = 3abc$, $a^2+2b^2+c^2 = 4abc$, $3a^2+2b^2+c^2 = 6abc$, $5a^2+b^2+c^2 = 5abc$

admit solutions with less than three singularities! I.e. $m_c = 1$ <u>and</u> $c = 1$.

The other 10 do not: $4a^2+2b^2+2c^2 = 8abc$,

$3a^2+3b^2+3c^2 = 9abc$, $6a^2+2b^2+c^2 = 6abc$, $4a^2+4b^2+2c^2 = 8abc$

$8a^2+b^2+c^2 = 4abc$, $6a^2+3b^2+c^2 = 6abc$, $9a^2+b^2+c^2 = 3abc$

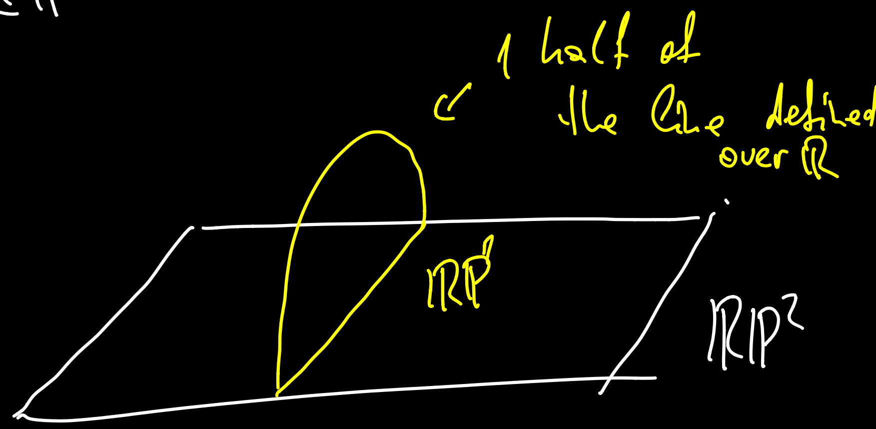
$8a^2+2b^2+c^2 = 4abc$, $5a^2+5b^2+c^2 = 5abc$, $6a^2+3b^2+2c^2 = 6abc$

$E(\tfrac{1}{2}, 2) \hookrightarrow \mathbb{CP}^2$

?

$\xrightarrow{\text{symp}}$

$\mathbb{P}(1,1,4)$



its complement is $E(\tfrac{1}{2}, 2)$